MATH 2060 TUT03

Thm 6.4.1 (Taylor's Thm)

Let
- $n \in \mathbb{N}$
- $f : [a,b] \to \mathbb{R}$ s.t. $(a < b)$
- $f', \ldots, f^{(n)}$ are cts on $[a,b]$ and
- $f^{(n+1)}$ exists on (a,b).

If $x_0 \in (a,b)$, then $\forall x \in (a,b)$, $\exists$ c between x_0 and x s.t.

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\substack{P_n(x) \\ n\text{-th Taylor's Polynomial of } f \text{ at } x_0}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\substack{R_n(x) \\ \text{remainder} \\ \text{(Lagrange form)}}}$$

§ 6.4

9. If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each fixed x_0 and x.

Ans: Note $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$ $\forall x \in \mathbb{R}$.
So g is indefinitely differentiable and satisfies the conditions of Taylor's Thm for all $n \in \mathbb{N}$.

Fix $x_0, x \in \mathbb{R}$.
The n-th remainder term in Taylor's Thm is
$R_n(x) = \frac{g^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1}$ for some c_n between x_0, x.

Since $g^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, we have

$$|R_n(x)| = \frac{|g^{n+1}(c_n)|}{(n+1)!}|x-x_0|^{n+1} \leq \frac{|x-x_0|^{n+1}}{(n+1)!} =: a_n$$

Want: $\lim(a_n) = 0$. Use Ratio Test!

Note $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{|x-x_0|}{n+1}\right)$ (assume $x_0 \neq x$)

$= 0 < 1$

By Ratio Test, $\lim_{n\to\infty}(a_n) = 0$
Therefore $\lim_{n\to\infty}(R_n(x)) = 0$ by Squeeze Thm. //

10. Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does not converge to zero as $n \to \infty$ for $x \neq 0$.

So h is smooth but not analytic ($\leftarrow$ locally given by a power series)

Ans: Clearly $h(x)$ is infinitely diff. for $x \neq 0$

Apply Leibniz' rule to $h'(x) = \frac{2}{x^3} e^{-1/x^2} = \frac{2}{x^3} h(x)$, we have

$$h^{(n+1)}(x) = \frac{d^n}{dx^n}\left(\frac{2}{x^3} h(x)\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{x^3}\right)^{(n-k)} h^{(k)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (2)(-3)(-4)\cdots(-(n-k+2)) x^{-(n-k+3)} h^{(k)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (n-k+2)! \frac{h^{(k)}(x)}{x^{n-k+3}} \qquad (*)$$

We prove by induction on $n \geq 0$ that

1) $\lim_{x \to 0} h^{(n)}(x)/x^m = 0 \qquad \forall m \in \mathbb{N}$

2) $h^{(n+1)}(0) = 0$.

When $n = 0$: 1) $\forall m \in \mathbb{N}$, by L'Hopital's rule,

$$\lim_{y \to +\infty} \frac{y^m}{e^y} = \lim_{y \to +\infty} \frac{m y^{m-1}}{e^y} = \cdots = \lim_{y \to +\infty} \frac{m!}{e^y} = 0$$

Let $y = 1/x^2$. Then $x \to 0 \Leftrightarrow y \to +\infty$.

Hence $\lim_{x \to 0} \frac{h(x)}{x^m} = \lim_{x \to 0} \frac{(1/x^2)^m}{e^{1/x^2}} \cdot x^m = 0$

2) $h'(0) = \lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = 0$.

Suppose 1), 2) are true for n.

Now, by (*),

$$\lim_{x\to 0} \frac{h^{(n+1)}(x)}{x^m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (n-k+2)! \left(\lim_{x\to 0} \frac{h^{(k)}(x)}{x^{n-k+3+m}} \right) = 0$$

Moreover, $h^{(n+2)}(0) = \lim_{x\to 0} \frac{h^{(n+1)}(x) - h^{(n+1)}(0)}{x-0} = 0$

This completes the induction.

Finally, $R_n(x) = h(x) - \sum_{k=0}^{n} \frac{\cancel{f^{(k)}(0)}^{\,0}}{k!} x^k = h(x)$

and so $\lim_{n\to\infty} R_n(x) = h(x) \neq 0$ for $x \neq 0$

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the ineq is clearly false when $x=0$.

11. If $x \in (0, 1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

$P_n(x)$

Use this to approximate $\ln 1.5$ with an error less than 0.01. Less than 0.001.

Ans: Let $f(t) = \ln(1+t)$.

Then f is infinitely diff. on $(-1, \infty)$ and

$$f^{(n)}(t) = \frac{(-1)^{n-1}(n-1)!}{(1+t)^n} \qquad t > -1,\ n \in \mathbb{N}.$$

Fix $x \in (0,1]$ and $n \in \mathbb{N}$.

By Taylor's Thm, $f(x) = P_n(x) + R_n(x)$,

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}$

and for some $c_n \in (0, x)$,

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!} x^{n+1} = \frac{1}{n+1} \cdot \frac{(-1)^n}{(1+c_n)^{n+1}} x^{n+1}$$

Now, $|R_n(x)| < \frac{1}{n+1} x^{n+1}$,

and so $\left| f(x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right) \right| = |R_n(x)| < \frac{1}{n+1} x^{n+1}$.

To approx $\ln(1.5)$, take $x = 0.5$.

Then $|\ln(1.5) - P_n(0.5)| < \frac{1}{n+1}(0.5)^{n+1}$

When $n=4$, $\frac{1}{n+1}(0.5)^{n+1} = 0.00625 < 0.01$.

So $\ln(1.5) \approx P_4(0.5) = \frac{77}{192}$ with error less than 0.01.

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15. Let f be continuous on $[a,b]$ and assume the second derivative f'' exists on (a,b). Suppose that the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a,b)$ such that $f''(c) = 0$.

Ans: By assumption,

$$f'(c_2) = \frac{f(b)-f(x_0)}{b-x_0} = \frac{f(x_0)-f(a)}{x_0-a} = f'(c_1)$$

Apply MVT to f on $[a, x_0]$, $\exists\, c_1 \in (a, x_0)$ s.t.

$$\frac{f(x_0)-f(a)}{x_0-a} = f'(c_1)$$

Apply MVT to f on $[x_0, b]$, $\exists\, c_2 \in (x_0, b)$ s.t.

$$\frac{f(b)-f(x_0)}{b-x_0} = f'(c_2)$$

Note $a < c_1 < c_2 < b$,

f'' exists on (a,b) $\Rightarrow$ f' cts and diff. on $[c_1, c_2]$.

Apply MVT again, $\exists\, c \in (c_1, c_2)$ s.t

$$f''(c) = \frac{f'(c_2)-f'(c_1)}{c_2-c_1} = 0$$

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Example Consider a function f whose second derivative $f''(x)$ exists and is continuous on $[0,1]$. Assume that $f(0)=f(1)=0$ and suppose that there exists $A>0$ such that $|f''(x)|\le A$ for $x\in[0,1]$. Show that

$$\left|f'\left(\frac{1}{2}\right)\right| \le \frac{A}{4} \quad \text{and} \quad |f'(x)| \le \frac{A}{2}$$

Ans: Apply Taylor's Thm to f with center $1/2$, we have $\forall x \in [0,1]$, $\exists\, c_x$ between $1/2$, x s.t

$$f(x) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(x-\tfrac{1}{2}) + \frac{f''(c_x)}{2!}(x-\tfrac{1}{2})^2$$

Put $x=1$:

$$0 = f(1) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(\tfrac{1}{2}) + f''(c_1)(\tfrac{1}{2^3}) \qquad ①$$

Put $x=0$:

$$0 = f(0) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(-\tfrac{1}{2}) + f''(c_0)(\tfrac{1}{2^3}) \qquad ②$$

① − ②: $\quad 0 = f'(\tfrac{1}{2}) + \tfrac{1}{8}(f''(c_1) - f''(c_0))$

$$\Rightarrow \quad |f'(\tfrac{1}{2})| \le \tfrac{1}{8}(|f''(c_1)| + |f''(c_0)|)$$
$$\le \tfrac{1}{8}(A + A) = \frac{A}{4}.$$

Fix $a \in (0,1)$. Apply Taylor's Thm to f with center a, we have $\exists\, c_0 \in (0,a)$, $\exists\, c_1 \in (a,1)$ s.t.

$$0 = f(1) = f(a) + f'(a)(1-a) + \frac{f''(c_1)}{2!}(1-a)^2 \qquad ③$$
$$0 = f(0) = f(a) + f'(a)(0-a) + \frac{f''(c_0)}{2!}(0-a)^2 \qquad ④$$

③ − ④: $\quad f'(a) + \tfrac{1}{2}[f''(c_1)(1-a)^2 + f''(c_0)a^2]$

$$\Rightarrow \quad |f'(a)| \le \frac{A}{2}[(1-a)^2 + a^2]$$
$$\le \frac{A}{2}[(1-a) + a] = \frac{A}{2}$$

If $a=0$, ③ $\Rightarrow$ $|f'(0)| = \tfrac{1}{2}|f''(c_1)| \le \frac{A}{2}$

If $a=1$, ④ $\Rightarrow$ $|f'(1)| = \tfrac{1}{2}|f''(c_0)| \le \frac{A}{2}$ //